JOURNAL OF APPROXIMATION THEORY 32, 96–102 (1981)

Interpolation and Best L_p Local Approximation

JERRY M. WOLFE

Department of Mathematics, University of Oregon, Eugene, Oregon 97403

Communicated by E. W. Cheney

Received June 1, 1979

Let S be a class of functions in $L_p[0, \delta_0]$, where $1 \leqslant p \leqslant \infty$ and $\delta_0 > 0$. Suppose that for each $\delta \in (0, \delta_0]$, a function $f \in L_p[0, \delta_0]$ has a best approximation $P_\delta(f)$ from S. The cluster points of the net $\{P_\delta(f)\}$ as $\delta \to 0^+$ are called the best local approximants of f (at 0) and the set of these will be denoted by $P_0(f)$. This idea is of particular interest in spline and piecewise polynomial approximation, where the behavior of approximants on small intervals is important.

The notion of best local approximation was introduced in [1] for the uniform norm and was continued in [2] for the L_2 case. Several questions were left unanswered and we shall attempt to answer some of these in this paper. In particular, by considering the problem from the more general point of view of interpolation we shall extend most of the results of [1, 2] to the case of general p and in addition we shall show that the set $P_0(f)$ is connected, at least if it is bounded. (This answers a question posed in [2].)

THE APPROXIMATING FAMILY

Let $1 \leqslant p \leqslant \infty$ be fixed and let $\{U_0,..., U_n\} \subset C^n[0, \delta_0]$ for some $\delta_0 > 0$ be such that the Wronskian matrix

$$W_n(U_0,..., U_n; x) \equiv \begin{bmatrix} U_0(x) & U_1(x) & \cdots & U_n(x) \\ U_0'(x) & U_1'(x) & \cdots & U_n'(x) \\ \vdots & \vdots & & \vdots \\ U_0^{(n)}(x) & U_1^{(n)}(x) & \cdots & U_n^{(n)}(x) \end{bmatrix}$$

satisfies $\det W_n(U_0,..., U_n; 0) \equiv \det W_n \neq 0$. Let H_n be the space spanned by $\{U_0,..., U_n\}$. Then H_n is an $n+1$ dimensional subspace of $L_p[0, \delta_0]$ and as noted in [2] the condition $\det W_n \neq 0$ implies that H_n is a Haar subspace on

0021-9045/81/060096–07$02.00/0

some subinterval $[0, \delta] \subset [0, \delta_0]$. Thus there is no harm in assuming at the outset that H_n is Haar on $[0, \delta_0]$.

Let $\{t_{i\nu}\}$ $i = 0,..., n$ be sequences satisfying

$$0 < t_{0\nu} < t_{1\nu} < \cdots < t_{n\nu} < \delta_0, \tag{1}$$

$$\lim_{\nu \to \infty} t_{i\nu} = 0, \qquad i = 0, 1,..., n, \tag{2}$$

and given $f \in C(0, \delta_0)$ let $P_\nu(f)$ be the unique element of H_n that inter polates f at each $t_{i\nu}$, $i = 0,..., n$. That is

$$P_\nu(t_{i\nu}) = f(t_{i\nu}), \qquad i = 0, 1,..., n, \quad \nu = 1, 2,.... \tag{3}$$

THEOREM 1. *Let $f \in C^k[0, \delta_0]$ and $0 \leqslant k \leqslant n$. Then every cluster point q of the sequence defined by* (3) *satisfies the condition that*

$$q^{(j)}(0) = f^{(j)}(0), \qquad j = 0,..., k. \tag{4}$$

In particular, if $k = n$ the sequence $\{P_\nu(f)\}$ converges (uniformly) to the unique $P_0 \in H_n$ that satisfies (4) *$k = n$.*

The proof of Theorem 1 requires the following two lemmas.

LEMMA 1. *For each ν, $P_\nu(f)(t) = \sum_{i=0}^n \alpha_{i\nu} U_i(t)$, where*

$$\alpha_{i\nu} = \frac{\begin{vmatrix} U_0(t_{0\nu}) & \cdots & U_{i-1}(t_{0\nu}) & f(t_{0\nu}) & U_{i+1}(t_{0\nu}) & \cdots & U_n(t_{0\nu}) \\ U_0(t_{1\nu}) & \cdots & U_{i-1}(t_{1\nu}) & f(t_{1\nu}) & U_{i+1}(t_{1\nu}) & \cdots & U_n(t_{1\nu}) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ U_0(t_{n\nu}) & \cdots & U_{i-1}(t_{n\nu}) & f(t_{n\nu}) & U_{i+1}(t_{n\nu}) & \cdots & U_n(t_{n\nu}) \end{vmatrix}}{\begin{vmatrix} U_0(t_{0\nu}) & \cdots & U_{i-1}(t_{0\nu}) & U_i(t_{0\nu}) & U_{i+1}(t_{0\nu}) & \cdots & U_n(t_{0\nu}) \\ U_0(t_{1\nu}) & \cdots & U_{i-1}(t_{1\nu}) & U_i(t_{1\nu}) & U_{i+1}(t_{1\nu}) & \cdots & U_n(t_{1\nu}) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ U_0(t_{n\nu}) & \cdots & U_{i-1}(t_{n\nu}) & U_i(t_{n\nu}) & U_{i+1}(t_{n\nu}) & \cdots & U_n(t_{n\nu}) \end{vmatrix}},$$

$$i = 0,..., n.$$

Proof. Cramer's rule.

LEMMA 2. *For each $i = 0,..., n$ the coefficients $\alpha_{i\nu}$ of $P_\nu(f)$ can be rewritten in the form*

$$\alpha_{i\nu} = \frac{\begin{vmatrix} U_0[t_{0\nu}] & \cdots & U_{i-1}[t_{0\nu}] & f[t_{0\nu}] & \cdots & U_n[t_{0\nu}] \\ U_0[t_{0\nu}, t_{1\nu}] & \cdots & U_{i-1}[t_{0\nu}, t_{1\nu}] & f[t_{0\nu}, t_{1\nu}] & \cdots & U_n[t_{0\nu}, t_{1\nu}] \\ \vdots & & \vdots & \vdots & & \vdots \\ U_0[t_{0\nu},\dots, t_{n\nu}] & \cdots & U_{i-1}[t_{0\nu},\dots, t_{n\nu}] & f[t_{0\nu},\dots, t_{n\nu}] & \cdots & U_n[t_{0\nu},\dots, t_{n\nu}] \end{vmatrix}}{\begin{vmatrix} U_0[t_{0\nu}] & \cdots & U_n[t_{0\nu}] \\ U_0[t_{0\nu}, t_{1\nu}] & \cdots & U_n[t_{0\nu}, t_{1\nu}] \\ \vdots & & \vdots \\ U_0[t_{0\nu},\dots, t_{n\nu}] & \cdots & U_n[t_{0\nu},\dots, t_{n\nu}] \end{vmatrix}},$$

where $g[s_0,\dots, s_j]$ *denotes the* $(j+1)$st *order divided difference of* g *with respect to* $s_0,\dots, s_j$.

Proof. This follows immediately from the properties of determinants and the fact that for each j,

$$g[s_0,\dots, s_j] = \frac{g(s_{j+1})}{(s_{j+1}-s_0)\cdots(s_{j+1}-s_j)} + \sum_{i=0}^{j} \alpha_i g[s_0,\dots, s_i],$$

where the coefficients α_i depend only on $s_0,\dots, s_i$. (Here we assume that the s_i's are distinct, of course.)

Proof of Theorem 1. If q is a cluster point of the sequence $\{P_\nu(f)\}$ then $q = \sum_{i=0}^{n} \alpha_i U_i(t)$ and there is a subsequence (which we do not relabel) such that

$$\lim_{\nu\to\infty} \alpha_{i\nu} = \alpha_i, \qquad i = 0,\dots, n.$$

But using Lemma 2 we see that α_i must have the form

$$\alpha_i = \frac{\begin{vmatrix} U_0(0) & \cdots & U_{i-1}(0) & f(0) & U_{i+1}(0) & \cdots & U_n(0) \\ U_0'(0) & \cdots & U_{i-1}'(0) & f'(0) & U_{i+1}'(0) & \cdots & U_n'(0) \\ \cdot & & \cdot & \vdots & \cdot & & \cdot \\ \cdot & & \cdot & \frac{f^{(k)}(0)}{k^1} & \cdot & & \cdot \\ \cdot & & \cdot & C_{k+1} & \cdot & & \cdot \\ \frac{U_0^{(n)}(0)}{n!} & \cdots & \frac{U_{i-1}^{(n)}(0)}{n!} & \vdots & \frac{U_{i+1}^{(n)}(0)}{n!} & \cdots & \frac{U_n^{(n)}(0)}{n!} \\ & & & C_n & & & \end{vmatrix}}{\begin{vmatrix} U_0(0) & \cdots & U_n(0) \\ \vdots & & \vdots \\ \frac{U_0^{(n)}(0)}{n!} & \cdots & \frac{U_n^{(n)}(0)}{n!} \end{vmatrix}},$$

$$i = 0,\dots, n, \qquad (5)$$

where $C_{k+1},\ldots, C_n$ are independent of i. But considering the interpolation problem of finding $\hat{q} \in H_n$ such that

$$\hat{q}^{(j)}(0) = f^{(j)}(0), \qquad j = 0,\ldots, k,$$
$$\hat{g}^{(j)}(0) = j!\, C_j, \qquad j = k+1,\ldots, n,$$

we see that it always has a unique solution and, in fact, solving by Cramer's rule yields the coefficients defined in (5). The convergence of the sequence $\{P_\nu(f)\}$ when $k = n$ is now clear. ∎

COROLLARY 1. *Let $f \in C^n[0, \delta_0]$ and let $1 \leqslant p \leqslant \infty$ be fixed. For each δ satisfying $0 < \delta \leqslant \delta_0$ let $P_\delta(f)$ denote the (unique) best L_p approximation to f on $[0, \delta]$ by elements of H_n. Then $P_\delta(f) \to P_0(f)$ uniformly on $[0, \delta_0]$ as $\delta \to 0$, where $P_0(f)$ is the unique element of H_n satisfying*

$$P_0^{(j)}(f) = f^{(j)}(0), \qquad j = 0,\ldots, n.$$

Proof. Since H_n is Haar of dimension $n+1$ it is well known that the error curve $f - P_\delta(f)$ must have at least $n+1$ zeros in $(0, \delta)$ so that $P_\delta(f)$ is uniquely defined by interpolation of f at any such set of $n+1$ points. Choosing $n+1$ such points for each δ and applying Theorem 1 we have the corollary. Corollary 1 generalizes Theorem 2.2 of [2] to the case of general values of p. The proof in [2] used asymptotic properties of gram matrices rather than the interpolation matrices of Theorem 1. Using a method similar to that employed in [2] we also have:

THEOREM 2. *Assume H_n is Haar of dimension $n+1$ on $[0, \delta_0]$. Then* $\det W_n \neq 0$ *if and only if the sequence $\{P_\nu(f)\}$ converges (uniformly on $[0, \delta_0]$) for every $f \in C^n[0, \delta_0]$. Here $\{P_\nu(f)\}$ is defined by* (4) *of Theorem* 1.

Proof. We have already seen that if $\det W_n \neq 0$ then $P_\nu(f) \to P_0(f)$ *uniformly on* $[0, \delta_0]$, *where* $P_0^{(j)}(f)(0) = f^{(j)}(0), j = 0,\ldots, n$.

Conversely, if $\{P_\nu(f)\}$ converges uniformly on $[0, \delta_0]$ to some $P_0(f)$ then for each ν, using Rolle's theorem there are points $\{\mathscr{E}_{j\nu}\}$, $j = 0, 1,\ldots, n$, such that $t_{0\nu} < \mathscr{E}_{j\nu} < t_{n\nu}$, and $P_\nu^{(j)}(f)(\mathscr{E}_{j\nu}) = f^{(j)}(\mathscr{E}_{j\nu})$, $j = 0, 1,\ldots, n$. Using the uniform convergences of $P_\nu^{(j)}(f)$ to $P_0^{(j)}(f)$, $j = 0,\ldots, n$, we conclude that $P_0^{(j)}(f)(0) = f^{(j)}(0), j = 0,\ldots, n$.

Thus the interpolation problem of finding $P \in H_n$ such that (*) $P^{(j)}(0) = f^{(j)}(0)$, $j = 0,\ldots, n$, has a solution for every $f \in C^n[0, \delta_0]$ and by examining the linear system coming from (*) when P is written in terms of the basis $\{U_0,\ldots, U_n\}$ it follows immediately that $\det W_n \neq 0$. ∎

TOPOLOGICAL PROPERTIES OF $P_0(f)$

Let $1 \leqslant p \leqslant \infty$ be fixed and let $f \in L_p[0, \delta_0]$, where we assume H_n is Haar on $[0, \delta_0]$ and $\det W_n \neq 0$ as before. In this section we study the set $P_0(f)$ of best local approximations (at 0). In [2] the authors gave examples showing that $P_0(f)$ may be empty, be uniquely defined (as we have already seen), or contain a continuum of points. They gave conditions under which $P_0(f)$ is convex in the case $p = 2$. We will now extend these results and prove the following result.

THEOREM 3. *Let $f \in L_p[0, \delta_0]$ be such that $P_0(f)$ is a bounded set. Then $P_0(f)$ is connected. Here if $p = 1$ or ∞ we assume that $f \in C[0, \delta_0]$. The analysis is based on the following lemma.*

LEMMA 3. *Let Φ: $(0, \delta] \to R^m$ be continuous. For each sequence $T = \{t_v\}_{v=1}^{\infty} \subset (0, \delta]$ with $t_v \to 0$, let A_T denote the set of limit points of $\{(t_v, \Phi_1(t_v),..., \Phi_m(t_v)\}$. Let B denote the set of all such convergent sequences and let $A = \bigcup_{T \in B} A_T$. Assume A is bounded. Then A is a connected subset of $\{(0, x_1,..., x_m) \mid x_i \in R, i = 1,..., m\} \subset R^{m+1}$.*

Proof. Clearly, we may assume that A has more than one element. Suppose A is not connected. Then there exist two distinct elements, say, $(0, x_1,..., x_m)$ and $(0, y_1,..., y_m)$, that are in different connected components of A, say, C_1 and C_2, respectively. Then there exist disjoint nonempty open sets U and V such that $A = (U \cap A) \cup (V \cap A)$ and such that $C_1 \subset U$ and $C_2 \subset V$. Also, without loss of generality, we may assume U is bounded (since A is). It follows that $\partial U \cap A$ is empty (where ∂S denotes the topological boundary of the set S) since an element in this set is not in $U \cap A$ or $V \cap A$. There exist sequences $T = \{t_v\}$ and $S = \{s_v\}$ such that

$$(t_v, \Phi_1(t_v),..., \Phi_m(t_v)) \to (0, x_1,..., x_m) \equiv x$$

and

$$(s_v, \Phi_1(s_v),..., \Phi_m(s_v)) \to (0, y_1,..., y_m) \equiv y.$$

Also by going to subsequences if necessary we may assume that $0 < s_{v+1} < t_v < s_v$ for all $v \geqslant 1$.

Let $\delta > 0$ be sufficiently small that $B_\delta(x) \subset U$ and $B_\delta(y) \subset V$ and let $\mathscr{E} = \delta/2$ (where $B_r(u)$ denotes the open ball of radius r centered at $u \in R^{m+1}$). Then there exists a $v_0 > 0$ such that for all $v \geqslant v_0$, $\max\{\|(t_v, \Phi_1(t_v),..., \Phi_m(t_v)) - x\|, \|(s_v, \Phi_1(s_v),..., \Phi_m(s_v) - y\|\} < \varepsilon$. Consider Φ on each of the intervals $I_v = [t_v, s_v]$ $v \geqslant v_0$. Now $\Phi(I_v)$ is connected and hence there exists a β_v with $t_v < \beta_v < s_v$ such that $(\beta_v, \Phi_1(\beta_v),..., \Phi_m(\beta_v)) \in \partial U$. That is, if this failed to occur, U and $W = \bar{U}^c$ would be disjoint nonempty open sets such that $\Phi(I_\nu) = (U \cap \Phi(I_\nu)) \cup (W \cap \Phi(I_\nu))$ (since $R^{m+1} = U \cup \partial U \cup (\bar{U})^c$ and all three sets are disjoint). Thus $\Phi(I_\nu)$ would not be connected—a contradiction. But ∂U is compact since U is bounded. Hence the sequence $(\beta_\nu, \Phi_1(\beta_\nu),..., \Phi_n(\beta_\nu))$ has a cluster point in ∂U which must be of the form $(0, w_1,..., w_m)$. Thus $\partial U \cap A$ is nonempty—a contradiction. Thus A is connected. ∎

COROLLARY 2. *Suppose that the hypotheses of Lemma 3 hold and in addition suppose that $\Phi_i \in C[0, \delta]$ for $i = 1,..., m$, $i \neq i_0$. Then A is convex.*

Proof. Every element of A is of the form $(0, \Phi_1(0),..., \Phi_{i_0-1}(0), x, \Phi_{i_0+1}(0),..., \Phi_m(0))$. Then A is a connected subset of $\{(0, \Phi_1(0),..., \Phi_{i_0-1}(0), v, \Phi_{i_0+1}(0),..., \Phi_m(0)) \mid v \in R\}$ and hence must be an "interval" and thus is convex. ∎

Proof of Theorem 3. Assume $P_0(f)$ is bounded. For each $0 < \delta < \delta_0$ the element $P_\delta(f) \in H_n$ that is the best approximation to f from H_n, in the L_p sense, is a continuous function of δ. Thus $P_\delta(f)$ may be written in the form $P_\varphi(f) = \sum_{i=0}^n a_i(\delta)\, U_i$, where each $a_i(\delta)$ is continuous on $(0, \delta_0]$. Since the correspondence $\sum_{i=0}^n a_i U_i \to (0, a_0,..., a_n)$ is a homeomorphism, the cluster points of $\{P_\delta(f)\}$ form a connected set if and only if the set of all cluster points of sequences of the form $\{\theta_\nu, (a_0(\theta_\nu),..., a_n(\theta_\nu))\}$ as $\theta_\nu \to 0^+$ form a connected set in R^{n+2}. But by Lemma 3, this set is connected, so $P_0(f)$ is connected. ∎

Theorem 3 and Corollary 2 immediately yield the following result, which generalizes a similar result in [2].

COROLLARY 3. *Let $f \in [0, \delta_0]$ be such that $f \in C^{n-1}[0, \delta_0]$. Then the set $P_0(f)$ is either empty or convex.*

Remark. Corollary 3 holds even if $P_0(f)$ is unbounded since it is easy to see that Corollary 2 is valid even if the set A is unbounded.

CONCLUDING REMARKS

The question of whether or not $P_0(f)$ is always convex is still open, even in the cases $p = 2$ and $p = \infty$. Also we conjecture that $P_0(f)$ is connected even if it is unbounded (if indeed that can happen) though this would not necessarily be true for the general types of functions considered in Lemma 3. Finally, the results in [2] concerning Pade approximation and quasi-rational approximation extend directly to the general setting of this paper and extensions to truly nonlinear families should be possible. This might be useful in studying nonlinear spline families [3].

REFERENCES

1. C. K. CHUI, O. SHISHA, AND P. W. SMITH, Best local approximation, *J. Approx. Theory* **15** (1975), 371–381.
2. C. K. CHUI, P. W. SMITH, AND J. D. WARD, Best local L_2 approximation, *J. Approx. Theory* **22** (1978), 254–261.
3. H. WERNER, Neure entwicklungen auf den Gebiete der nichtlinearen splinefunktionen, *Z. Angew. Math. Mech.* **58** (1978), 86–95.